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Critical Exponents of Quasilinear Parabolic Equations

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In this paper we study the critical exponents of the Cauchy problem in $\mathbf{R}^n$ of the quasilinear singular parabolic equations: $u_t = \operatorname{div}(|\nabla u|^{m-1}\nabla u) + t^s|x|^\sigma u^p$, with non-negative initial data. Here $s \geq 0, (n-1)/(n+1) < m < 1, p > 1$ and $\sigma > n(1-m) - (1+m+2s)$. We prove that $p_c \equiv m + (1+m+2s+\sigma)/n > 1$ is the critical exponent. That is, if $1 < p \leq p_c$ then every non-trivial solution blows up in finite time, but for $p > p_c$, a small positive global solution exists. © 2002 Elsevier Science (USA)

Key Words: quasilinear parabolic equations; critical exponents.

1. INTRODUCTION

The study of blow-up for nonlinear parabolic equations probably originates from Fujita [7, 8], where he studied the following Cauchy problem of semilinear heat equation

$$\begin{aligned} u_t &= \triangle u + u^p, \quad x \in \mathbf{R}^n, \quad t > 0, \\ u(x,0) &= u_0(x) \geq 0, \quad x \in \mathbf{R}^n, \end{aligned} \tag{1.1}$$

where $p > 1$ and obtained the following result:

(a) If $1 < p < 1 + 2/n$, then every non-trivial solution of (1.1) blows up in finite time;

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(b) If $p > 1 + 2/n$ and $u_0(x) \le \delta e^{-|x|^2}$ for some $0 < \delta \ll 1$, then (1.1) admits a global solution.

For the critical case $p = 1 + 2/n$, it was shown by Hayakawa [11] for dimensions $n = 1, 2$, Kobayashi *et al.* [13], and Aronson and Weinberger [2] for all $n \ge 1$ that (1.1) possesses no global solution $u(\dot{x}, t)$ satisfying

$$\|u(\cdot, t)\|_\infty < \infty \quad \text{for } t \ge 0.$$

Weissler [26] proved that if $p = 1 + 2/n$, then (1.1) possesses no global solution $u(x, t)$ satisfying

$$\|u(\cdot, t)\|_q < +\infty \quad \text{for } t \ge 0 \quad \text{and some} \quad q \in [1, +\infty).$$

The value $p_c = 1 + 2/n$ is called the critical exponent of (1.1). It plays an important role in studying the behavior of the solution to (1.1).

These elegant works revealed a new phenomenon of nonlinear PDEs and stimulated the study of similar features for various nonlinear evolution equations. Especially, the following Cauchy problems of porous medium equations

$$\begin{aligned} u_t &= \triangle u^m + t^s |x|^\sigma u^p, \quad x \in \mathbf{R}^n, \quad t > 0, \\ u(x, 0) &= u_0(x) \ge 0, \qquad\qquad x \in \mathbf{R}^n \end{aligned} \tag{1.2}$$

were studied by many authors [3, 10, 14–20, 23, 25], where $m > (n-2)_+/n$, $s \ge 0$, and $\sigma > -1$ if $n = 1$ or $\sigma > -2$ if $n \ge 2$ and $p > \max\{m, 1\}$. Recently, Qi [23] proved the following: If $p_c \triangleq m + (m-1)s + (2 + \sigma + 2s)/n > 1$ then p_c is the critical exponent of (1.2); i.e., when $1 < p \le p_c$ every non-trivial solution of (1.2) blows up in finite time, and when $p > p_c$, (1.2) admits small positive global solution.

The following Cauchy problems of the quasilinear degenerate parabolic equation

$$\begin{aligned} u_t &= \operatorname{div}(|\nabla u|^{m-1} \nabla u) + u^p, \quad x \in \mathbf{R}^n, \quad t > 0, \\ u(x, 0) &= u_0(x) \ge 0, \qquad\qquad x \in \mathbf{R}^n \end{aligned} \tag{1.3}$$

with $m > 1$, were studied by the authors of [9, 10, 20, 22]. They obtained that $p_c \triangleq m + (1+m)/n$ is the critical exponent of (1.3) and p_c belongs to the blow-up case ([10, 20]).

In this paper, we shall consider the following general quasilinear "singular" parabolic equations

$$\begin{aligned} u_t &= \operatorname{div}(|\nabla u|^{m-1} \nabla u) + t^s |x|^\sigma u^p, \quad x \in \mathbf{R}^n, \quad t > 0, \\ u(x, 0) &= u_0(x) \ge, \not\equiv 0, \qquad\qquad x \in \mathbf{R}^n, \end{aligned} \tag{I}$$

where $(n-1)/(n+1) < m < 1$, $s \ge 0$, $p > 1$ and $\sigma > n(1-m) - (1 + m + 2s)$, $u_0(x)$ is a continuous function in $\mathbf{R}^n$.

By a solution of (I) we mean a continuous function $u : \mathbf{R}^n \times (0, T) \to \mathbf{R}^+$ with $Du \in L^1_{loc}(0, T; L^1_{loc}(\mathbf{R}^n))$, and Eq. (I) is satisfied in the sense of distribution in $\mathbf{R}^n \times (0, T)$, where $T > 0$ is the maximal existence time. Since $m > (n-1)/(n+1)$ and $u_0(x)$ are non-negative and continuous functions in $\mathbf{R}^n$, the existence, uniqueness, and comparison principle of solutions to (I) had been proved in [5]. Moreover, the following result holds:

PROPOSITION 1. *If $u_0(x)$ is a non-trivial and non-negative continuous function, then the solution $u(x, t)$ of* (I) *is positive in $\mathbf{R}^n$ for any $t > 0$, $Du \in C^\alpha_{loc}(\mathbf{R}^n \times (0, T))$ for some $0 < \alpha < 1$.*

The main purpose of the present paper is to find the critical exponent of (I) *and prove that every non-trivial solution blows up in finite time if $p > 1$ is less than or equal to the critical exponent. In this context, we say that $u(x, t)$ blows up in finite time $T > 0$ if*

$$w(t) = \int_\Omega u(x, t)\, dx \to +\infty \quad as \quad t \to T^-$$

for a finite $T > 0$, where Ω is a bounded domain in $\mathbf{R}^n$. It follows easily from Proposition 1 *that this definition is the same as that of Friedman and McLeod* [6].

Our main result reads as follows:

THEOREM 1. *Assume that $(n-1)/(n+1) < m < 1$, $s \geq 0$, $p > 1$ and $\sigma > n(1-m) - (1+m+2s)$. Then $p_c = m + (1+m+2s+\sigma)/n > 1$ is the critical exponent of* (I). *That is, if $1 < p \leq p_c$ then every non-trivial solution of* (I) *blows up in finite time; if $p > p_c$ then* (I) *has small non-trivial global solutions.*

This paper is organized as follows. In Section 2 we discuss the qualitative behaviors and give some estimates of solutions to the homogeneous problem

$$\begin{aligned} u_t &= \operatorname{div}(|\nabla u|^{m-1}\nabla u), \quad x \in \mathbf{R}^n, \quad t > 0, \\ u(x, 0) &= u_0(x) \geq, \not\equiv 0, \qquad x \in \mathbf{R}^n. \end{aligned} \tag{II}$$

In Section 3, for convenience, we first discuss the special case of (I): $s = 0$, i.e.,

$$\begin{aligned} u_t &= \operatorname{div}(|\nabla u|^{m-1}\nabla u) + |x|^\sigma u^p, \; x \in \mathbf{R}^n, \quad t > 0, \\ u(x, 0) &= u_0(x) \geq, \not\equiv 0, \qquad x \in \mathbf{R}^n, \end{aligned} \tag{III}$$

and prove that if $1 < p \leq \tilde{p}_c \triangleq m + (1+m+\sigma)/n$ then every non-trivial solution of (III) blows up in finite time. In Section 4 we prove Theorem 1.

Our method is similar in nature to that in [23].

Remark. We end this section with a simple but very useful reduction. Since, by Proposition 1, the solution $u(x, t)$ of (I) is continuous and positive in $\mathbf{R}^n$ for any $t > 0$, we may assume, without loss of generality, that $u_0(x)$ is continuous and positive in $\mathbf{R}^n$. By the comparison principle we need only consider that $u_0(x)$ is radially symmetric and non-increasing; i.e., $u_0(x) = u_0(r)$ with $r = |x|$ and $u_0(r)$ non-increasing in r. Therefore, the solution $u(x, t)$ of (I) is also radially symmetric and non-increasing in $r = |x|$.

2. ESTIMATES OF SOLUTIONS TO (II)

In this section we discuss (II) for the radially symmetric case. The main results are the two propositions.

PROPOSITION 2. *Assume that* $(n-1)/(n+1) < m < 1$ *and* $u_0(x)$ *is a non-trivial and non-negative continuous function. If, in addition,* $u_0(x)$ *is a radially symmetric and non-increasing function, then the solution* $u(x, t)$ *of* (II) *satisfies*

$$u_t \geq -\frac{\alpha}{t} u \quad \text{for all} \quad x \in \mathbf{R}^n \quad \text{and} \quad t > 0, \tag{2.1}$$

where $\alpha = n/[1 + m - n(1-m)] > 0$.

PROPOSITION 3. *Under the assumptions of Proposition* 2,

$$u(x, t) \geq \delta(t-\varepsilon)^{-\alpha}(1 + C|x|^{(1+m)/m}(t-\varepsilon)^{-\beta})^{-m/(1-m)} \tag{2.2}$$

for $|x| \geq 1, t > \varepsilon > 0$, *where* δ *and* C *are positive constants and* $\beta = (1 + m)/[m(1 + m - n(1-m))] > 0$.

Proof of Proposition 2. By the uniqueness we know that $u(x, t) = u(r, t)$ is radially symmetric and non-increasing in r, $r = |x|$. Let

$$v = \frac{m}{m-1} u^{(m-1)/m},$$

then we have $v < 0$, $v' < 0$ (where $v' = \frac{\partial v}{\partial r}$) and

$$\begin{aligned} \operatorname{div}(|\nabla u|^{m-1}\nabla u) &= u \operatorname{div}(|\nabla v|^{m-1}\nabla v) + u^{-1}|\nabla u|^{m+1} \\ &\geq u \operatorname{div}(|\nabla v|^{m-1}\nabla v), \\ v_t &= \frac{m-1}{m} v \operatorname{div}(|\nabla v|^{m-1}\nabla v) + |\nabla v|^{m+1}. \end{aligned} \tag{2.3}$$

To prove (2.1) it is sufficient to prove that $\operatorname{div}(|\nabla v|^{m-1}\nabla v) \geq -\alpha/t$ by (2.3). Denote $w = \operatorname{div}(|\nabla v|^{m-1}\nabla v)$ and let $z = -v$, we find $z' > 0$ and

$$\begin{aligned} z_t &= -\frac{m-1}{m} z \operatorname{div}(|\nabla z|^{m-1}\nabla z) - |\nabla z|^{m+1} \\ &= -\frac{m-1}{m} z\left(m(z')^{m-1}z'' + \frac{n-1}{r}(z')^m\right) - (z')^{m+1} \\ &= \frac{m-1}{m} zw - (z')^{m+1}, \\ w &= -\left(m(z')^{m-1}z'' + \frac{n-1}{r}(z')^m\right). \end{aligned} \tag{2.4}$$

By direct computation we have

$$\begin{aligned} -w_t &= m(z')^{m-1}z_t'' + m(m-1)(z')^{m-2}z_t'z'' + m\frac{n-1}{r}(z')^{m-1}z_t', \\ z_t' &= \frac{m-1}{m}z'w + \frac{m-1}{m}zw' - (m+1)(z')^m z'', \\ z_t'' &= \frac{m-1}{m}z''w + 2\frac{m-1}{m}z'w' + \frac{m-1}{m}zw'' \\ &\quad - (m+1)(z')^m z''' - m(m+1)(z')^{m-1}(z'')^2. \end{aligned}$$

By a series of calculations we have

$$\begin{aligned} -w_t &= (m-1)z(z')^{m-1}\Delta w + m(m+1)(z')^{m-1}z''w \\ &\quad + (1-m)w^2 + 2(m-1)(z')^m w' + (m-1)^2 zz''(z')^{m-2}w' \\ &\quad + m(1-m^2)(z')^{2m-2}(z'')^2 - m(m+1)(z')^{2m-1}z'''. \end{aligned} \tag{2.5}$$

It follows from (2.4) that

$$\begin{aligned} -w' &= m(z')^{m-1}z''' + m(m-1)(z')^{m-2}(z'')^2 \\ &\quad - \frac{n-1}{r^2}(z')^m + m\frac{n-1}{r}(z')^{m-1}z''. \end{aligned}$$

Substituting the above into (2.5) we get

$$\begin{aligned} -w_t &= (m-1)a(r,t)\Delta w + b(r,t)w' + (1-m)w^2 + m(m+1)z''(z')^{m-1}w \\ &\quad + m(m+1)\frac{n-1}{r}(z')^{2m-1}z'' - (m+1)\frac{n-1}{r^2}(z')^{2m}, \end{aligned}$$

where $a(r,t), b(r,t)$ are functions produced by $z(r,t)$ and $a(r,t) > 0$. By use of (2.4),

$$\begin{aligned} -w_t &= (m-1)a(r,t)\Delta w + b(r,t)w' - 2mw^2 \\ &\quad - 2(m+1)\frac{n-1}{r}(z')^m w - n(m+1)\frac{n-1}{r^2}(z')^{2m}. \end{aligned}$$

Taking into account the Cauchy inequality

$$-2(z')^m w \le (z')^{2m} + w^2,$$

we find

$$\begin{aligned} -w_t &\le (m-1)a(r,t)\triangle w + b(r,t)w' - 2mw^2 + (m+1)\frac{n-1}{n}w^2 \\ &= (m-1)a(r,t)\triangle w + b(r,t)w' - \frac{1+m-n(1-m)}{n}w^2; \end{aligned}$$

i.e.,

$$w_t \ge (1-m)a(r,t)\triangle w - b(r,t)w' + \frac{1}{\alpha}w^2.$$

Let $y(r,t) = -\alpha/t$. It is obvious that $y_t = (1-m)a(r,t)\triangle y - b(r,t)y' + (1/\alpha)y^2$. Since $y(r,0) = -\infty$, it follows by the comparison principle that $w \ge -\alpha/t$; i.e., $\operatorname{div}(|\nabla v|^{m-1}\nabla v) \ge -\alpha/t$. ∎

Remark. The above proof is similar to an argument of Aronson and Benilan [1] of the porous media equation $u_t = \triangle u^m$. But, it seems to us that there is no direct way of transforming their result to the present case.

To prove Proposition 3 we first state a comparison lemma which can be proved by using the methods of [5, Chap. 6] or [12].

LEMMA 2.1. *Let* $0 \le \tau < +\infty$ *and* $S = \{x \in \mathbf{R}^n, |x| > 1\} \times [\tau, +\infty)$. *Assume that* v, w *are non-negative functions and satisfy*

$$\begin{aligned} &v_t = \operatorname{div}(|\nabla v|^{m-1}\nabla v), \quad w_t = \operatorname{div}(|\nabla w|^{m-1}\nabla w) \quad \textit{in } S, \\ &v(x,t) \le w(x,t), \quad |x| = 1, \quad \tau < t < +\infty, \\ &v(x,\tau) \le w(x,\tau), \quad |x| \ge 1. \end{aligned}$$

Then

$$v(x,t) \le w(x,t) \quad \textit{in } S.$$

Proof of Proposition 3. Since $(n-1)/(n+1) < m < 1$, by the results of [24] we have that problem (II) has the similarity solutions

$$U_\mu(x,t) = \mu^\theta U(\mu x, t), \ \ \theta = (1+m)/(1-m),$$

where $\mu > 0$ is a parameter,

$$U(x,t) = U_1(x,t) = t^{-\alpha}[1 + b|x|^{(1+m)/m}t^{-\beta}]^{-m/(1-m)},$$

β is given in the statement Proposition 3, and $b = \frac{1-m}{1+m}(1+m-n(1-m))^{-1/m}$. Since $U(x,t)$ can be written as

$$U(x,t) = t^{-\alpha+m\beta/(1-m)}[t^\beta + b|x|^{(1+m)/m}]^{-m/(1-m)},$$

and $-\alpha + m\beta/(1-m) = 1/(1-m) > 0$, we see that

$$U_\mu(x, t-\varepsilon) = 0 \quad \text{for} \quad |x| \geq 1, \ t = \varepsilon, \quad \text{where} \quad 0 < \varepsilon \ll 1. \tag{2.6}$$

By Proposition 2,

$$u(1, t) \geq \varepsilon^\alpha u(1, \varepsilon) t^{-\alpha} \quad \text{for all} \quad t \geq \varepsilon. \tag{2.7}$$

Now we estimate $U_\mu(1, t-\varepsilon)$. When $t/(t-\varepsilon) \geq K > 1$, it follows that $t \leq K\varepsilon/(K-1)$, $t - \varepsilon \leq \varepsilon/(K-1)$. Therefore,

$$u(1, t) \geq \varepsilon^\alpha u(1, \varepsilon) t^{-\alpha} \geq (K/(K-1))^{-\alpha} u(1, \varepsilon), \tag{2.8}$$

and

$$\begin{aligned} U_\mu(1, t-\varepsilon) &= \mu^\theta (t-\varepsilon)^{-\alpha + m\beta/(1-m)} ((t-\varepsilon)^\beta + b\mu^{(1+m)/m})^{-m/(1-m)} \\ &\leq \mu^\theta (t-\varepsilon)^{-\alpha + m\beta/(1-m)} b^{-m/(1-m)} \mu^{-\theta} \\ &= b^{-m/(1-m)} (t-\varepsilon)^{1/(1-m)} \\ &\leq b^{-m/(1-m)} \varepsilon^{1/(1-m)} (K-1)^{-1/(1-m)} \\ &\leq (K/(K-1))^{-\alpha} u(1, \varepsilon) \end{aligned} \tag{2.9}$$

if K is suitably large, say $K \geq K_0$, for some $K_0 \gg 1$ independent of μ.

When $t/(t-\varepsilon) \leq K$,

$$\begin{aligned} U_\mu(1, t-\varepsilon) &= \mu^\theta (t-\varepsilon)^{-\alpha} [1 + b\mu^{(1+m)/m} (t-\varepsilon)^{-\beta}]^{-m/(1-m)} \\ &\leq \mu^\theta (t-\varepsilon)^{-\alpha} \leq \mu^\theta K^\alpha t^{-\alpha} \\ &\leq \varepsilon^\alpha u(1, \varepsilon) t^{-\alpha} \end{aligned} \tag{2.10}$$

if $\mu > 0$ is suitably small. Combining (2.7)–(2.10) we see that

$$U_\mu(1, t-\varepsilon) \leq u(1, t) \quad \text{for all} \quad t \geq \varepsilon. \tag{2.11}$$

Equations (2.6) and (2.11), when combined with Lemma 2.1, yield

$$u(x, t) \geq U_\mu(x, t-\varepsilon) \quad \text{for all} \quad |x| \geq 1 \quad \text{and} \quad t \geq \varepsilon.$$

Consequently (2.2) holds. ∎

3. THE SPECIAL CASE $s = 0, 1 < p \leq \tilde{p}_c$

In this section we study problem (III) and prove a blow-up result.

THEOREM 2. *Let m, p, and σ be as in Theorem 1. If $1 < p \leq \tilde{p}_c = m + (m+1+\sigma)/n$, then every non-trivial solution of* (III) *blows up in finite time.*

Let $\psi(x)$ be a smooth, radially symmetric and non-increasing function which satisfies

$$0 \leq \psi(x) \leq 1, \psi(x) \equiv 1 \quad \text{for} \quad |x| \leq 1 \quad \text{and} \quad \psi(x) \equiv 0 \quad \text{for} \quad |x| \geq 2.$$

Let $\psi_0(x)$ be a smooth radially symmetric and non-decreasing function which satisfies

$$0 \leq \psi_0(x) \leq 1, \psi_0(x) \equiv 0 \quad \text{for} \quad |x| \leq 1 \quad \text{and} \quad \psi_0(x) \equiv 1 \quad \text{for} \quad |x| \geq 2.$$

Set $\psi_\ell(x) = \psi(x/\ell)$. It follows that for $\ell \geq 1$, $\psi_\ell(x)$ is a smooth, radially symmetric, and non-increasing function which satisfies

$$0 \leq \psi_\ell(x) \leq 1, \psi_\ell(x) \equiv 1 \quad \text{for} \quad |x| \leq \ell \quad \text{and} \quad \psi_\ell(x) \equiv 0 \quad \text{for} \quad |x| \geq 2\ell.$$

Denote $\phi_\ell(x) = \psi_0(x)\psi_\ell(x)$. Then $\phi_\ell(x)$ is a smooth and radially symmetric function and satisfies for $\ell > 2$,

$$\begin{gathered} 0 \leq \phi_\ell(x) \leq 1, \phi_\ell(x) \equiv 0 \qquad \text{for } |x| \leq 1, \\ \phi_\ell(x) \equiv 1 \text{ for } \; 2 \leq |x| \leq \ell \quad \text{and} \quad \phi_\ell(x) \equiv 0 \text{ for } \; |x| \geq 2\ell. \end{gathered}$$

Moreover, $\phi_\ell(x)$ is non-decreasing for $1 \leq |x| \leq 2$ and non-increasing for $\ell \leq |x| \leq 2\ell$.

Denote

$$w_\ell(t) = \int_\Omega u(x,t)\phi_\ell(x)\,dx,$$

where $\Omega = \mathbf{R}^n \backslash B_{1/2}$ with $B_{1/2}$ being the ball with radial $1/2$ and center at the origin. By the remark at the end of Section 1, we may assume, without loss of generality, u is radially symmetric. Then w_ℓ is an increasing function of ℓ and

$$\begin{aligned} \frac{d}{dt}\int_\Omega u\phi_\ell\,dx &= -\int_\Omega |\nabla u|^{m-1}\nabla u \cdot \nabla \phi_\ell + \int_\Omega |x|^\sigma \phi_\ell(x) u^p\,dx \\ &= -\omega_n \int_1^{2\ell} |u'|^{m-1}u'\phi_\ell' r^{n-1}\,dr + \int_\Omega |x|^\sigma \phi_\ell(x)u^p\,dx \\ &= \omega_n \int_1^{2\ell} |u'|^m \phi_\ell' r^{n-1}\,dr + \int_\Omega |x|^\sigma \phi_\ell(x)u^p\,dx \\ &\geq -\omega_n \int_1^{2\ell} |u'|^m |\phi_\ell'| r^{n-1}\,dr + \int_\Omega |x|^\sigma \phi_\ell(x)u^p\,dx, \end{aligned} \tag{3.1}$$

where ω_n is the area of the unit sphere in $\mathbf{R}^n$.

$$\int_1^{2\ell} |u'|^m |\phi'_\ell| r^{n-1}\,dr = \int_1^{2\ell} r^{m(n-1)}|u'|^m|\phi'_\ell|^m|\phi'_\ell|^{1-m} r^{(1-m)(n-1)}\,dr$$
$$\le \left(\int_1^{2\ell} r^{n-1}|u'||\phi'_\ell|\,dr\right)^m \left(\int_1^{2\ell} |\phi'|r^{n-1}\,dr\right)^{1-m}. \quad (3.2)$$

$$\begin{aligned}\int_1^{2\ell} r^{n-1}|u'||\phi'_\ell| &= \int_1^2 r^{n-1}|u'||\phi'_\ell|\,dr + \int_\ell^{2\ell} r^{n-1}|u'||\phi'_\ell|\,dr\\ &= -\int_1^2 r^{n-1}u'\phi'_\ell\,dr + \int_\ell^{2\ell} r^{n-1}u'\phi'_\ell\,dr\\ &= -\frac{1}{\omega_n}\int_{\Omega_1}\nabla u\cdot\nabla\phi_\ell\,dx + \frac{1}{\omega_n}\int_{\Omega_2}\nabla u\cdot\nabla\phi_\ell\,dx\\ &= \frac{1}{\omega_n}\int_{\Omega_1} u\triangle\phi_\ell\,dx - \frac{1}{\omega_n}\int_{\Omega_2} u\triangle\phi_\ell\,dx\\ &\le \frac{1}{\omega_n}\int_\Omega |u\triangle\phi_\ell|\,dx, \end{aligned} \quad (3.3)$$

where $\Omega_1 = \{x \mid 1 \le |x| \le 2\}$, $\Omega_2 = \{x \mid \ell \le |x| \le 2\ell\}$.

$$\int_1^{2\ell} r^{n-1}|\phi'_\ell|\,dr = \int_{1/\ell}^2 \ell^{n-1} r^{n-1}|\phi'|\,dr \le C\ell^{n-1}, \quad (3.4)$$

where $\phi(x) = \psi(x)\psi_0(x)$. It is easy to see that C is independent of ℓ. Substituting (3.2)–(3.4) *into* (3.1) *we obtain*

$$\frac{d}{dt}\int_\Omega u\phi_\ell\,dx \ge -C\ell^{(1-m)(n-1)}\left(\int_\Omega u|\triangle\phi_\ell|\,dx\right)^m + \int_\Omega |x|^\sigma\phi_\ell u^p\,dx. \quad (3.5)$$

By use of the Hölder inequality we get

$$\int_\Omega u|\triangle\phi_\ell|\,dx \le \left(\int_\Omega |x|^{-(k-1)\sigma}|\triangle\phi_\ell|^k\phi_\ell^{-(k-1)}\,dx\right)^{1/k}$$
$$\times\left(\int_\Omega |x|^\sigma\phi_\ell u^p\,dx\right)^{1/p}, \quad (3.6)$$

where $k = p/(p-1)$. Thus, we have by (3.5)

$$\frac{d}{dt}\int_\Omega u\phi_\ell\,dx \ge -C\ell^{(1-m)(n-1)}C(\phi_\ell,\ell)^{m/k}\left(\int_\Omega |x|^\sigma\phi_\ell u^p\,dx\right)^{m/p}$$
$$+\int_\Omega |x|^\sigma\phi_\ell u^p\,dx, \quad (3.7)$$

where

$$C(\phi_\ell, \ell) = \int_\Omega |x|^{-\sigma/(p-1)} |\triangle\phi_\ell|^{p/(p-1)} \phi_\ell^{-1/(p-1)}\, dx \le C\ell^{[(p-1)n-(\sigma+2p)]/(p-1)}.$$

Therefore

$$\frac{d}{dt}\int_\Omega u\phi_\ell\, dx \ge \left\{-C\ell^{(n-m-1)-m(n+\sigma)/p} + \left(\int_\Omega |x|^\sigma \phi_\ell u^p dx\right)^{(p-m)/p}\right\} \times \left(\int_\Omega |x|^\sigma \phi_\ell u^p\, dx\right)^{m/p}. \tag{3.8}$$

By Hölder's inequality we have

$$\int_\Omega |x|^\sigma \phi_\ell u^p\, dx \ge \left(\int_\Omega u\phi_\ell\, dx\right)^p \left(\int_\Omega |x|^{-\sigma/(p-1)}\phi_\ell\, dx\right)^{-(p-1)}.$$

Hence

$$\int_\Omega |x|^\sigma \phi_\ell u^p\, dx \ge \begin{cases} Cw_\ell^p \ell^{\sigma-n(p-1)} & \text{if } \sigma < n(p-1), \\ Cw_\ell^p (\log \ell)^{-(p-1)} & \text{if } \sigma = n(p-1), \\ Cw_\ell^p & \text{if } \sigma > n(p-1). \end{cases} \tag{3.9}$$

Next, we quote a result from [23].

LEMMA 3.1 [23]. *Let $w(t)$ be a positive continuous function which satisfies the inequality*

$$\frac{dw}{dt} \ge Cw^q$$

in the distributional sense, where $C > 0$ is a constant and $q > 1$. Then $w(t)$ is an increasing function, and there exists a finite $T > 0$ such that $w(t) \to +\infty$ as $t \to T^-$.

Proof of Theorem 2. First we consider the case $\sigma < n(p-1)$. It follows from (3.8) and (3.9) that

$$\frac{dw_\ell}{dt} \ge \left\{ -C_1\ell^{(n-m-1)-m(n+\sigma)/p} + C_2 w_\ell^{p-m}(t)\ell^{[\sigma-n(p-1)](p-m)/p}\right\} \times \left(\int_\Omega |x|^\sigma \phi_\ell u^p\, dx\right)^{m/p}. \tag{3.10}$$

(a) $p < \tilde{p}_c = m + (m+1+\sigma)/n$.

Under this assumption, $[\sigma - n(p-1)](p-m)/p > n-m-1-m(n+\sigma)/p$, and consequently

$$\frac{\ell^{[\sigma-n(p-1)](p-m)/p}}{\ell^{(n-m-1)-m(n+\sigma)/p}} \longrightarrow +\infty \quad \text{as} \quad \ell \to +\infty. \tag{3.11}$$

Using the fact that w_ℓ is an increasing function of ℓ, we find from (3.10) and (3.11) that there exists $\ell \gg 1$ and $\delta > 0$ such that

$$\frac{dw_\ell}{dt} \geq \delta w_\ell^p(t)\ell^{\sigma-n(p-1)} \quad \text{for all} \quad t > 0.$$

Thus w_ℓ, and consequently u, blows up in finite time by Lemma 3.1.

(b) $p = \tilde{p}_c = m + (m+1+\sigma)/n.$

In this case $[\sigma - n(p-1)](p-m)/p = n - m - 1 - m(n+\sigma)/p$. If we can prove that for any $M > 0$, there exists $l > 0$ and $t > 0$ such that

$$\int_\Omega u(x,t)\phi_\ell(x)\,dx > M,$$

then it can be shown just as in the above case that w_ℓ, and hence u, blows up in finite time. Otherwise, $u(\cdot,t) \in L^1(\Omega)$ for all $t > 0$ and there exists an $M_0 > 0$ such that

$$\|u(t)\|_{L^1(\Omega)} \leq M_0 \quad \text{for all} \quad t > 0. \tag{3.12}$$

We prove (3.12) is impossible. Suppose the contrary; it is clear from (3.6) that if $\int_\Omega |x|^\sigma u^p\,dx < +\infty$ then

$$\int_\Omega u|\triangle\phi_\ell|\,dx \to 0 \quad \text{and} \quad \ell^{(1-m)(n-1)}\left(\int_\Omega u|\triangle\phi_\ell|\,dx\right)^m \to 0 \text{ as } \ell \to +\infty,$$

because $n - m - 1 - m(n+\sigma)/p < 0$. By (3.5) we get $w_\ell'(t) \geq \frac{1}{2}\int_\Omega |x|^\sigma \times\phi_\ell u^p\,dx$. If $\int_\Omega |x|^\sigma u^p\,dx = +\infty$, then $w_\ell'(t) \geq 1$ by (3.7). Hence

$$w_\ell'(t) \geq \kappa_\ell(t) \triangleq \min\left\{1, \frac{1}{2}\int_\Omega |x|^\sigma \phi_\ell u^p\,dx\right\}, \quad \ell \gg 1.$$

Upon integration we have

$$w_\ell(t) - w_\ell(0) \geq \int_0^t \kappa_\ell(\tau)\,d\tau.$$

Let $w(t) = \int_\Omega \psi_0(x)u(x,t)\,dx$, and take $\ell \to +\infty$ in the above inequality to obtain

$$w(t) - w(0) \geq \int_0^t \kappa(\tau)\,d\tau, \tag{3.13}$$

where $\kappa(t) = \min\{1, \frac{1}{2}\int_\Omega |x|^\sigma \psi_0(x)u^p\,dx\}$.

Using (2.2), by direct computation we have

$$\begin{aligned}\int_\Omega |x|^\sigma\psi_0(x)u^p\,dx &\geq \delta^p(t-\varepsilon)^{-1}\int_{|y|\geq(t-\varepsilon)^{-\theta}} |y|^\sigma\psi_0(y(t-\varepsilon)^\theta)\\ &\qquad\times(1+C|y|^{(1+m)/m})^{-mp/(1-m)}\,dy\\ &\geq C(t-\varepsilon)^{-1} \quad \text{as} \quad t \gg 1,\end{aligned} \tag{3.14}$$

where $\theta = 1/[1+m-n(1-m)] > 0$. In view of (3.13) and (3.14) it yields

$$\lim_{t\to+\infty} w(t) = +\infty; \quad \text{i.e.,} \quad \lim_{t\to+\infty}\int_\Omega \psi_0(x)u(x,t)\,dx = +\infty.$$

Since $\psi_0(x) \le 1$, this shows that (3.12) is impossible. And hence $u(x,t)$ blows up in finite time.

Next, we consider the case $\sigma \ge n(p-1)$. Since $m > (n-1)/(n+1)$, it follows that $n-m-1-m(n+\sigma)/p < 0$.

For the case $\sigma = n(p-1)$, combining (3.8) and (3.9), we find

$$\begin{aligned}\frac{dw_\ell}{dt} &\ge \left\{-C\ell^{(n-m-1)-m(n+\sigma)/p} + Cw_\ell^{p-m}(\log\ell)^{(m-p)(p-1)/p}\right\}\\ &\quad\times\left(\int_\Omega |x|^\sigma \phi_\ell u^p\,dx\right)^{m/p}.\end{aligned}\tag{3.15}$$

Using the fact that w_ℓ is an increasing function of ℓ, we find from (3.15) that there exist $\ell \gg 1$ and $\delta > 0$ such that

$$\frac{dw_\ell}{dt} \ge \delta(\log\ell)^{1-p}w_\ell^p(t) \quad \text{for} \quad t > 1.$$

Thus w_ℓ, and consequently u, blows up in finite time.

The case of $\sigma > n(p-1)$ can be handled similarly using the third inequality of (3.9). This completes the proof of Theorem 2. ∎

4. PROOF OF THEOREM 1

In this section we shall prove Theorem 1 for the general case (I).

When $1 < p \le p_c = m + (m+1+2s+\sigma)/n$, using the methods similar to those of the last section and [23], it can be proved that every non-trivial solution of (I) blows up in finite time. We omit the details.

When $p > p_c$, we shall prove that (I) has global positive solutions for small initial data. By the comparison principle, it is enough to prove this conclusion for the following problem (since $s \ge 0$)

$$\begin{aligned} u_t &= \operatorname{div}(|\nabla u|^{m-1}\nabla u) + (1+t)^s|x|^\sigma u^p, && x\in\mathbf{R}^n,\ t>0,\\ u(x,0) &= u_0(x) \ge 0, && x\in\mathbf{R}^n,\end{aligned}\tag{4.1}$$

where the constants m, s, σ, and p are as in problem (I). We shall show the existence of global solutions of (4.1) by constructing global similarity solutions. They take the form

$$u(x,t) = (1+t)^{-\alpha}w(r) \qquad \text{with} \quad r = |x|(1+t)^{-\beta},$$

where $\alpha = [(1+m)(1+s)+\sigma]/K$, $\beta = [p-1+(1-m)(1+s)]/K$, $K = (p-1)(1+m)-\sigma(1-m)$. It is easy to verify that α, β, and K satisfy the following relations:

$$m(\alpha+\beta)+\beta = \alpha+1 = p\alpha-\beta\sigma-s, \quad K > 0.$$

The resulting ODE for w is

$$\begin{aligned} &m|w'|^{m-1}w'' + \frac{n-1}{r}|w'|^{m-1}w' + \alpha w + \beta r w' + r^\sigma w^p = 0, \quad r > 0, \\ &w(0) = \eta > 0, |w'|^{m-1}w'(0) = \lim_{r\to 0^+}(-r^{\sigma+1}w^p(r)/(n+\sigma)). \end{aligned} \tag{4.2}$$

We observe that a function $\bar{u}(x,t) = (1+t)^{-\alpha}v(|x|(1+t)^{-\beta})$ is an upper solution of (4.1) if and only if $v(r)$ satisfies the inequality

$$m|v'|^{m-1}v'' + \frac{n-1}{r}|v'|^{m-1}v' + \alpha v + \beta r v' + r^\sigma v^p \le 0, \quad r > 0. \tag{4.3}$$

We first discuss the case $\sigma \ge 0$. In this case, we try to find an upper solution of (4.1), i.e., the solution of (4.3). Let

$$v(r) = \varepsilon(1+br^k)^{-q},$$

where $k = (1+m)/m$, $q = m/(1-m)$, and ε and b are positive constants to be determined later. By direct computation we have

$$\begin{aligned} v' &= -\varepsilon\, q\, b\, k\, r^{k-1}(1+br^k)^{-q-1}, \\ v'' &= \varepsilon\, q(q+1)b^2\, k^2\, r^{2k-2}(1+br^k)^{-q-2} - \varepsilon\, q\, b\, k(k-1)r^{k-2}(1+br^k)^{-q-1}. \end{aligned}$$

$v(r)$ satisfies (4.3) if and only if

$$\begin{aligned} &\varepsilon\, q\, b\, k[\varepsilon^{m-1}(q\, b\, k)^m - \beta]r^k(1+b\, r^k)^{-q-1} \\ &\quad + \varepsilon(\alpha - n\,\varepsilon^{m-1}(q\, b\, k)^m)(1+b\, r^k)^{-q} + \varepsilon^p r^\sigma(1+b\, r^k)^{-pq} \le 0. \end{aligned} \tag{4.4}$$

By $p > p_c$ we find $\sigma + q(1-p)k = \sigma + (1-p)(1+m)/(1-m) < 0$. It follows that there exists $a > 0$ such that

$$r^\sigma(1+b\, r^k)^{q(1-p)} \le a \qquad \text{for all} \qquad r \ge 0 \tag{4.5}$$

since $\sigma \ge 0$. Choose $b = b(\varepsilon)$ such that

$$\varepsilon^{m-1}(q\, b\, k)^m = \beta; \text{ i.e.,} \quad b = \beta^{1/m}\varepsilon^{(1-m)/m}(qk)^{-1}.$$

For this choice of b, (4.4) is equivalent to

$$\alpha - n\beta + \varepsilon^{p-1}r^\sigma(1+br^k)^{q-pq} \le 0. \tag{4.6}$$

By (4.5) we see that (4.6) is true if the following inequality holds

$$\alpha - n\beta + a\varepsilon^{p-1} \le 0. \tag{4.7}$$

In view of $p > p_c = m + (m+1+2s+\sigma)/n$ it follows that $\alpha < n\beta$. Hence, there exists $\varepsilon_0 > 0$ such that (4.7) holds for all $0 < \varepsilon \le \varepsilon_0$. These arguments show that $v(r) = \varepsilon(1+b(\varepsilon)r^k)^{-q}$ satisfies (4.3) for all $0 < \varepsilon \le \varepsilon_0$. Using the comparison principle we get that the solution $u(x,t)$ of (4.1) exists globally provided that $u(x,0) \le v(|x|)$. And hence, so does the solution of (I).

Next, we consider the case $\sigma < 0$. For this case, it will be proved that (4.2) has a ground state for small η. By a standard argument we can prove that for any given $\eta > 0$, there exists a unique solution w of (4.2), which is twice continuously differentiable where $w'(r) \ne 0$; see [21, 22]. Denote $R(\eta)$ the maximum of R for which $w(r) > 0$ in $[0, R)$. So, $0 < R(\eta) \le +\infty$, and $w(R(\eta)) = 0$ when $R(\eta) < +\infty$.

We divide the proof into several lemmas.

LEMMA 4.1. *The solution* $w(r)$ *of* (4.2) *satisfies* $w'(r) < 0$ *in* $(0, R(\eta))$. *In addition, if* $R(\eta) = +\infty$ *then* $w(r) \to 0$ *as* $r \to +\infty$.

Proof. We first prove $w'(r) < 0$ for $0 < r < R(\eta)$. When $\sigma + 1 \le 0$, we have that $|w'|^{m-1}w'(0) = \lim_{r\to 0^+}(-r^{\sigma+1}w^p(r)/(n+\sigma)) < 0$. Therefore $w'(r) < 0$ for $r \ll 1$. If there exists $r_0 : 0 < r_0 < R(\eta)$ such that $w'(r) < 0$ in $(0, r_0)$ and $w'(r_0) = 0$, then $(|w'|^{m-1}w')'(r_0) \ge 0$. But, by Eq. (4.2) we see that $(|w'|^{m-1}w')'(r_0) = -(\alpha w(r_0) + r_0^\sigma w^p(r_0)) < 0$, a contradiction.

When $\sigma + 1 > 0$, it follows that $w'(0) = 0$. Using Eq. (4.2) one has

$$n(|w'|^{m-1}w')'(0) = -\alpha w(0) - \lim_{r\to 0^+} r^\sigma w^p < 0.$$

Hence $|w'|^{m-1}w'(r) < 0$, and consequently $w'(r) < 0$ for $r \ll 1$. Similar to the case of $\sigma + 1 \le 0$ it follows that $w'(r) < 0$ for all $0 < r < R(\eta)$.

If $R(\eta) = +\infty$. Since $w'(r) < 0$ and $w(r) > 0$ in $(0, +\infty)$, $\lim_{r\to+\infty} w(r) = L$. If $L > 0$, integration of (4.2) gives

$$r^{n-1}(|w'|^{m-1}w' + \beta r w) = -\int_0^r \{(\alpha - n\beta)s^{n-1}w(s) + s^{n+\sigma-1}w^p(s)\}\,ds. \tag{4.8}$$

Let $r \to +\infty$ in (4.8) we find

$$\lim_{r\to+\infty} \frac{|w'|^{m-1}w'}{r} = -\frac{\alpha}{n}L - \frac{A}{n},$$

where $A = L^p$ when $\sigma = 0$, $A = 0$ when $\sigma < 0$, and $A = +\infty$ when $\sigma > 0$. It follows that

$$\lim_{r\to+\infty} w'(r) = -\infty,$$

a contradication. Thus $w(r) \to 0$ as $r \to +\infty$. ∎

LEMMA 4.2. *For any given small* $\eta > 0$ *there exists* $R_0(\eta) > 0$, *which satisfies* $\lim_{\eta\to 0^+} R_0(\eta) = +\infty$ *and such that*

$$w(r) > 0,\ |w'|^{m-1}w'(r) + \beta\, r\, w(r) > 0 \qquad \text{on } [2, R_0(\eta)]. \tag{4.9}$$

Proof. Let $z = \eta - w$. Then $z'(r) = -w'(r) > 0$, $0 < z(r) < \eta$ and $z(r)$ satisfies

$$\begin{aligned} &((z')^m)' + \frac{n-1}{r}(z')^m + \beta\, r\, z' = \alpha(\eta - z) + r^\sigma(\eta - z)^p, \quad r > 0, \\ &\qquad z(0) = 0,\, |z'|^{m-1}z'(0) = \lim_{r\to 0}(r^{\sigma+1}(\eta - z)^p/(n+\sigma)). \end{aligned} \tag{4.10}$$

Integration of (4.10) gives

$$\begin{aligned} r^{n-1}(z')^m + \beta\, r^n z &= \int_0^r [(n\,\beta - \alpha)s^{n-1}z(s) + \alpha\,\eta\, s^{n-1}]\, ds \\ &\quad + \int_0^r s^{n+\sigma-1}(\eta - z)^p\, ds \\ &\le \frac{\alpha\eta}{n} r^n + \left(\beta - \frac{\alpha}{n}\right) r^n z(r) + \frac{1}{n+\sigma}\, \eta^p r^{n+\sigma}, \end{aligned} \tag{4.11}$$

since $p > p_c$ implies $n\,\beta > \alpha$. Let $R_0(\eta)$ be the first value of r, where $z(r) = \eta - \eta^{(p+1)/2}$. Then $R_0(\eta) > 0$ and $z(r) \le \eta - \eta^{(p+1)/2} < \eta$ for all $0 < r \le R_0(\eta)$. From (4.11) it follows that for $0 < r \le R_0(\eta)$

$$\begin{aligned} r^{n-1}(z')^m &< \frac{\alpha\,\eta}{n}\, r^n + \left(\beta - \frac{\alpha}{n}\right) \eta\, r^n + \frac{1}{n+\sigma}\, \eta^p r^{n+\sigma} \\ &= \beta\,\eta\, r^n + \frac{1}{n+\sigma}\, \eta^p r^{n+\sigma}; \end{aligned}$$

i.e.,

$$(z')^m < \beta\,\eta\, r + \frac{1}{n+\sigma}\, \eta^p r^{\sigma+1}.$$

Since $m < 1$, it follows that

$$z'(r) < \left(\beta\,\eta\, r + \frac{1}{n+\sigma}\eta^p r^{\sigma+1}\right)^{1/m} \le C(\eta^{1/m}r^{1/m} + \eta^{p/m}r^{(\sigma+1)/m}).$$

Integrating this inequality from 0 to $R_0(\eta)$ we have

$$\eta \le \eta^{(p+1)/2} + C(\eta^{1/m}(R_0(\eta))^{(1+m)/m} + \eta^{p/m}(R_0(\eta))^{(m+\sigma+1)/m}).$$

In view of $p > 1$ and $m < 1$, it is obvious that $R_0(\eta) \to +\infty$ as $\eta \to 0^+$. And $z(r) \le \eta - \eta^{(p+1)/2}$, $z(R_0(\eta)) = \eta - \eta^{(p+1)/2}$; consequently $w(r) \ge \eta^{(p+1)/2}$ and $w(R_0(\eta)) = \eta^{(p+1)/2}$, for all $0 \le r \le R_0(\eta)$.

Integration of (4.2) gives, for $0 \le r \le R_0(\eta)$,

$$\begin{aligned} &r^{n-1}|w'|^{m-1}w' + \beta\, r^n w(r) \\ &\quad = \int_0^r (n\,\beta - \alpha)\, s^{n-1}w(s)\, ds - \int_0^r s^{n+\sigma-1}w^p(s)\, ds \\ &\quad \ge (n\,\beta - \alpha)\, w\,(R_0(\eta)) \int_0^r s^{n-1}\, ds - \eta^p \int_0^r s^{n+\sigma-1}\, ds \\ &\quad = \left(\beta - \frac{\alpha}{n}\right) \eta^{(p+1)/2} r^n - \frac{1}{n+\sigma}\, \eta^p r^{n+\sigma} \\ &\quad = \eta^{(p+1)/2} r^n \left(\beta - \frac{\alpha}{n} - \frac{1}{n+\sigma}\, \eta^{(p-1)/2} r^\sigma\right). \end{aligned}$$

Since $\sigma < 0$, $p > 1$ and $\eta \ll 1$ (note $n + \sigma > 0$, $n\beta > \alpha$), it follows that

$$r^{n-1}|w'|^{m-1}w' + \beta r^n w(r) > 0 \quad \text{for} \quad 2 \le r \le R_0(\eta).$$

The proof of Lemma 4.2 is completed. ∎

Now we prove that, for the case $\sigma < 0$, (4.2) has a ground state for small η. Choose $\eta_0 : \eta_0^{p-1} < n\beta - \alpha$ and such that (4.9) holds for all $0 < \eta \le \eta_0$. Integrating (4.2) from $R_0(\eta)$ to $r(R_0(\eta) < r < R(\eta))$ gives

$$\begin{aligned}
&r^{n-1}|w'|^{m-1}w' + \beta r^n w(r) \\
&\quad = (r^{n-1}|w'|^{m-1}w' + \beta r^n w(r))|_{r=R_0(\eta)} \\
&\qquad + (n\beta - \alpha)\int_{R_0(\eta)}^{r} s^{n-1}w(s)\,ds - \int_{R_0(\eta)}^{r} s^{n+\sigma-1}w^p(s)\,ds \\
&\quad > \int_{R_0(\eta)}^{r} s^{n-1}w(s)[n\beta - \alpha - s^{\sigma}w^{p-1}(s)]\,ds \\
&\quad \ge \int_{R_0(\eta)}^{r} s^{n-1}w(s)(n\beta - \alpha - \eta^{p-1})\,ds > 0
\end{aligned} \tag{4.12}$$

since $\sigma < 0$, $R_0(\eta) > 2$, and $w(s) < \eta$. In view of $w(r) > 0$ and $w'(r) < 0$ for $0 < r < R(\eta)$, it follows that $R(\eta) = +\infty$ by (4.12). Therefore (4.2) has a ground state. ∎

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